

# CHARACTER THEORY FOR SEMISIMPLE HOPF ALGEBRAS

SEBASTIAN BURCIU

**ABSTRACT.** We study the induction and restriction functor from a Hopf subalgebra of a semisimple Hopf algebra. The image of the induction functor is described when the Hopf subalgebra is normal. In this situation, at the level of characters this image is isomorphic to the image of the restriction functor. A criterion for subcoalgebras to be invariant under the adjoint action is given generalizing Masuoka's result for normal Hopf algebras.

## INTRODUCTION

The representation and character theory of semisimple Hopf algebras over an algebraically closed field of characteristic 0 has been developed in the last thirty years. Many results analogous to the classical theory of finite groups were obtained. One of the most important results was Zhu's proof that the character ring of a semisimple Hopf algebra is a semisimple ring [13]. The character theory for coalgebras started in [5] and then continued in [11], [12], as well as in other papers. An important updated reference on results on character theory and its applications in classification of finite Hopf algebras can be found in [10].

In this paper we study the induction and restriction functor from a Hopf subalgebra of a semisimple Hopf algebra. Important results are obtained when the Hopf subalgebra is normal. In this situation, at the level of characters, the images of the induction map and restriction map are isomorphic as algebras. Recently was proven that a Hopf subalgebra is normal if and only if it is a depth two subalgebra [3]. One proof of this result uses the character theory for normal Hopf subalgebras developed in [2]. This paper continues the study began in [2].

Our approach uses the commuting pair described in [14]. We show that the Fourier transform of a semisimple Hopf algebra  $H$  is a morphism of  $D(H)$ -modules. As in [14] an important role is played by

the module obtained by inducing the trivial module from  $H$  to the Drinfeld double  $D(H)$ .

The paper is organized as follows. In the first section we recall basic results about semisimple Hopf algebras and its character ring.

In Section 2 we study co-normal ideals of a semisimple Hopf algebra  $H$ . We show that a two sided ideal of a semisimple Hopf algebra  $H$  is co-normal if and only if its character as left module is central in  $H^*$ . This follows by duality from a description of all subcoalgebras of  $H$  that are invariant under the adjoint action. More precisely, we prove that a subcoalgebra of  $H$  is invariant under the adjoint action of  $H$  if and only if its character as left (or right)  $H$ -comodule is central in  $H$ . This generalizes a well known result of Masuoka stating that a Hopf subalgebra of  $H$  is normal if and only if its integral is central in  $H$  [8]. The proof uses the Drinfeld double  $D(H)$  and the commuting pair of modules from [14].

In the third section we prove few results about the induction functor from a semisimple Hopf subalgebra  $K$  of  $H$  to  $H$ . This functor induces a map at the level of character rings  $\text{ind} : C(K) \rightarrow C(H)$ . Its image is an ideal inside  $C(H)$ . We also have an algebra map  $\text{res} : C(H) \rightarrow C(K)$  induced by the restriction functor. In the last section we consider the situation when  $K$  is a normal Hopf subalgebra of  $H$ . We prove that the images of the restriction and induction maps are isomorphic as algebras. Another description of this image is given at the end of this section.

We work over an algebraically closed field  $k$  of characteristic zero. For a vector space  $V$  by  $|V|$  is denoted the dimension of  $V$ . All the modules are supposed to be left modules. The comultiplication, counit and antipode of a Hopf algebra are denoted by  $\Delta$ ,  $\epsilon$  and  $S$ , respectively. We use Sweedler's notation  $\Delta(x) = \sum x_1 \otimes x_2$  for all  $x \in H$ . All the other notations for Hopf algebras are those used in [9]

## 1. PRELIMINARIES

**1.1. Theory of characters.** Throughout this paper  $H$  will be a finite dimensional semisimple Hopf algebra over the algebraically closed field  $k$ . Then  $H$  is also cosemisimple and  $S^2 = \text{Id}$  (see [6]). We use the notation  $\Lambda_H \in H$  for the idempotent integral of  $H$  ( $\epsilon(\Lambda_H) = 1$ ) and  $t_H \in H^*$  for the idempotent integral of  $H^*$  ( $t_H(1) = 1$ ). Denote by  $\text{Irr}(H)$  the set of irreducible characters of  $H$  and let  $C(H)$  be the character ring of  $H$ . Then  $C(H)$  is a semisimple subalgebra of  $H^*$  [13] and  $C(H) = \text{Cocom}(H^*)$ , the space of cocommutative elements of  $H^*$ .

By duality, the character ring of  $H^*$  is a semisimple subalgebra of  $H$  and under this identification it follows that  $C(H^*) = \text{Cocom}(H)$ .

If  $M$  is an  $H$ -module with character  $\chi$  then  $M^*$  is also an  $H$ -module with character  $\chi^* = \chi \circ S$ . This induces an involution “ $*$ ” :  $C(H) \rightarrow C(H)$  on  $C(H)$ .

If  $\text{Irr}(H)$  is the set of irreducible  $H$ -modules then from [9] it follows that the regular character of  $H$  is given by the formula

$$(1.1) \quad |H|t_H = \sum_{\chi \in \text{Irr}(H)} \chi(1)\chi.$$

The dual formula is

$$(1.2) \quad |H|\Lambda_H = \sum_{d \in \text{Irr}(H^*)} \epsilon(d)d$$

One also has  $t_H(\Lambda_H) = \frac{1}{|H|}$  [6]. There is an associative nondegenerate bilinear form on  $C(H)$  given by

$$(\chi, \mu) = \chi\mu(\Lambda_H).$$

It follows that  $(\chi, \mu) = m_H(\chi, \mu^*)$  where  $m_H$  is the multiplicity form on  $C(H)$ . For two modules  $M$  and  $N$  of  $H$  one has  $m_H(\chi_M, \chi_N) = \dim_k \text{Hom}_H(M, N)$ . The pairs  $\{\chi\}_{\chi \in \text{Irr}(H)}$  and  $\{\chi^*\}_{\chi \in \text{Irr}(H)}$  form dual bases with respect to  $(, )$ .

A Hopf subalgebra of  $H$  is called normal if  $\sum h_1 x S h_2 \in K$  for all  $h \in H$  and  $x \in K$ .

**1.2. Properties of the function  $\mathcal{F}$ .** Let  $H$  be a finite dimensional semisimple Hopf algebra. Then  $H^*$  is a left and right  $H$ -module via  $(a \rightharpoonup f)(b) = f(ba)$  and  $(f \leftharpoonup a)(b) = f(ab)$ . Similarly  $H$  is an  $H^*$  left and right  $H$ -module via  $f \rightharpoonup a = \sum f(a_2)a_1$  and  $a \leftharpoonup f = \sum f(a_1)a_2$ .

Let  $\mathcal{F} : H \rightarrow H^*$  be the map given by  $\mathcal{F}(a) = a \rightharpoonup t_H$  for all  $a \in H$ . It is well known that  $\mathcal{F}$  is bijective [9]. The inverse of  $\mathcal{F}$  is given by

$$\mathcal{F}^{-1}(f) = |H|Sf \rightharpoonup \Lambda_H.$$

**Proposition 1.3.** *Let  $K$  be a Hopf subalgebra of  $H$ . Then*

$$(1.4) \quad H^* = \mathcal{F}(K) \oplus K^\perp.$$

*Proof.* let  $i : K \hookrightarrow H$  be the canonical inclusion of  $K$  in  $H$ . Since  $i^* : H^* \rightarrow K^*$  is a surjective Hopf map one has  $i^*(t_H) = t_K$ . On the other hand the map  $i^*$  is just restriction to  $K$ . It follows that  $t_H \downarrow_K^H = t_K$ . Suppose that  $f \in \mathcal{F}(K) \cap K^\perp$ . Then  $f = a \rightharpoonup t_H$  for some  $a \in K$ . Then  $0 = f(x) = t_H(ax) = t_K(ax)$  for all  $x \in K$ . Nondegeneracy of  $t_K$  [7] implies  $a = 0$ .  $\square$

The map  $\mathcal{F}$  sends the center  $\mathcal{Z}(H)$  of  $H$  into the character ring  $C(H)$  of  $H$  [9].

**Proposition 1.5.** *Let  $K$  be a Hopf subalgebra of  $H$ . Then the following diagram*

$$\begin{array}{ccc} H & \xrightarrow{\mathcal{F}_H} & H^* \\ i \uparrow & & \downarrow i^* \\ K & \xrightarrow{\mathcal{F}_K} & K^*. \end{array}$$

*is commutative where  $i$  is the canonical inclusion.*

*Proof.* As before  $i^*(t_H) = t_K$  where  $i^*$  is just restriction map from  $H$  to  $K$ . If  $a \in K$  then

$$i^*(\mathcal{F}_H(i(a))) = i^*(a \rightharpoonup t_H) = a \rightharpoonup t_K = \mathcal{F}_K(a)$$

□

## 2. THE COMMUTING PAIR OF MODULES

Let  $H$  be a finite dimensional semisimple Hopf algebra. For  $f \in H^*$  and  $h \in H$  define  $f \rightharpoonup h = \sum f(h_{(2)})h_{(1)}$  and  $h \leftarrow f = \sum f(h_{(1)})h_{(2)}$ .

The Drinfeld double  $D(H)$  of  $H$  is defined as follows:  $D(H) \cong H^{*cop} \otimes H$  as coalgebras. The multiplication on  $D(H)$  is given by:

$$(g \otimes h)(f \otimes l) = \sum g(h_1 \rightharpoonup f \leftarrow S^{-1}h_3) \otimes h_2l.$$

Its antipode is given by  $S(f \otimes h) = S(h)S^{-1}(f)$ . If  $H$  is finite dimensional semisimple over  $k$  then  $D(H)$  is also semisimple and cosemisimple [9].

Consider the induced module from  $H$  to  $D(H)$  given by  $A_0 = D(H) \otimes_H k$ . Then  $A_0 \cong H^*$  where the action is given by  $a.f(x) = f(\sum S^{-1}a_2xa_1)$  and  $g.f = gf$  for all  $a, x \in H$  and  $f, g \in A^*$ . [4]

The module  $A_0$  can also be realized on  $H$  as following:  $x.a = \sum x_1aS(x_2)$  and  $f.x = x \leftarrow S^{-1}f = \sum f(S^{-1}x_1)x_2$ .

It was proven by Zhu [14] that

$$\text{End}_{D(H)}(A_0) = C(H)^{op}$$

and the isomorphism is given by  $\chi \mapsto R_\chi$  where  $R_\chi$  is right multiplication by  $\chi$  on  $H^*$ . Therefore the map

$$C(H) \rightarrow \text{End}_{D(H)}(H^*), \quad \chi \mapsto R_{S(\chi)}$$

is a ring isomorphism. Also the map

$$C(H) \rightarrow \text{End}_{D(H)}(H), \quad \chi \mapsto (S(\chi) \rightharpoonup -)$$

is a ring isomorphism.

Let  $E_1, E_2, \dots, E_s$  be the set of central primitive idempotents of  $C(H)$ . From the above facts it follows that the homogeneous  $D(H)$ -components of  $H^*$  are given by  $H^*E_i$  for all  $1 \leq i \leq s$ .

**2.1. The map  $\mathcal{F}$ .** Let  $\text{Irr}(H^*)$  be the set of irreducible characters of  $H^*$ . For any  $d \in \text{Irr}(H^*)$  let  $\xi_d \in H^*$  the primitive idempotent corresponding to  $d$ . It can be checked that  $\mathcal{F}(d) = \frac{1}{\epsilon(d)}\xi_{d^*}$  where  $d^* = S(d)$  (see also [9] for the dual version.) It is also known that  $\mathcal{F}$  sends the algebra  $C(H^*)$  to the center  $\mathcal{Z}(H^*)$  of  $H^*$ .

**Proposition 2.1.** *The map  $\mathcal{F}$  is an isomorphism of  $D(H)$ -modules.*

*Proof.* It has to be verified that  $\mathcal{F}(a.h) = a.\mathcal{F}(h)$  and  $\mathcal{F}(f.h) = f.\mathcal{F}(h)$  for all  $a, h \in H$  and  $f \in H^*$ . One has that  $\mathcal{F}(a.h) = \mathcal{F}(\sum a_1 h S(a_2)) = \sum a_1 h S(a_2) \rightarrow t_H$ . Thus  $\mathcal{F}(a.h)(l) = t_H(\sum a_1 h S(a_2)l)$  for  $l \in H$ . On the other hand  $(a.\mathcal{F}(h))(l) = \mathcal{F}(h)(\sum S a_2 l a_1) = \sum t_H(S a_2 l a_1 h)$ . Since  $|H|t_H$  is the regular character of  $H$  it follows that  $t_H(xy) = t_H(yx)$  for all  $x, y \in H$ . Thus  $\mathcal{F}(a.h) = a.\mathcal{F}(h)$ .

For the other equality one has  $\mathcal{F}(f.h)(l) = \sum f(Sh_1)(\mathcal{F}(h_2)(l)) = \sum f(Sh_1)t_H(lh_2)$ . On the other hand  $(f.\mathcal{F}(h))(l) = \sum f(l_1)\mathcal{F}(h)(l_2) = \sum f(l_1)t_H(l_2h)$ . In order to prove that  $\mathcal{F}(f.h) = f.\mathcal{F}(h)$  we will prove that  $\sum Sh_1 t_H(lh_2) = \sum l_1 t_H(l_2h)$  for all  $l, h \in H$ . Indeed  $\sum l_1 t_H(l_2h) = \sum l_1 h_2 t_H(l_2 h_3) S(h_1) = \sum t_H(lh_2) S(h_1)$ . We have used that  $t_H$  is an integral and therefore  $\sum a_1 t_H(a_2) = t_H(a)1$  for all  $a \in H$ .  $\square$

For a coalgebra  $C$  let  $\text{Irr}(C)$  be the set of irreducible  $H^*$ -characters contained in  $C$ . Then  $d_C = \sum_{d \in \text{Irr}(C)} \epsilon(d)d$  is the character of  $C$  as left (or right)  $H$ -comodule.

**Remark 2.2.** *Let  $C$  be a subcoalgebra of  $H$ . Then*

$$\mathcal{F}(C) = \oplus_{d \in \text{Irr}(C)} H^* \xi_d.$$

*Indeed it is enough to verify this equality for a simple subcoalgebra  $C$  with character  $d$ . This follows since  $C = d \leftarrow H^*$  and  $\mathcal{F}$  is an isomorphism of  $D(H)$ -modules.*

**2.2. Co-normal ideals.** A vector subspace  $I$  of  $H$  is called co-normal if

$$\sum S v_3 v_1 \otimes v_2 \in H \otimes I$$

for all  $v \in I$ . (usually  $I$  will be an ideal.)

Let  $I$  be an ideal of  $H$  and  $\pi : H \rightarrow H/I$  the canonical projection. Then  $(H/I)^*$  is a subcoalgebra of  $H^*$  via  $\pi^*$ .

**Proposition 2.3.** *Let  $I$  be an ideal of  $H$ . Then  $I$  is a co-normal ideal if and only if the subcoalgebra  $(H/I)^*$  is invariant under the adjoint action  $H^*$  on itself.*

*Proof.* Note that  $(H/I)^* = I^\perp$  inside  $H^*$ .  $(H/I)^*$  is invariant under the adjoint action of  $H^*$  if and only if  $\sum f_1 g S(f_2) \in I^\perp$  for all  $f \in H^*$  and  $g \in I^\perp$ . This is equivalent to  $(\sum f_1 g S(f_2))(x) = 0$  for all  $f \in H^*$ ,  $g \in I^\perp$  and  $x \in I$ . But  $(\sum f_1 g S(f_2))(x) = f(\sum x_1 S(x_3) g(x_2))$  which implies that  $\sum x_1 S(x_3) g(x_2) = 0$  for all  $g \in I^\perp$  and  $x \in I$ . This is equivalent to  $\sum x_1 S(x_3) \otimes x_2 \in H \otimes I$ .  $\square$

The following lemma will be used in the proof of next theorem.

**Lemma 2.4.** *Let  $A$  be a finite dimensional semisimple algebra over an algebraically closed field of characteristic zero. Let  $x \in A$  be an idempotent of  $A$  and  $e \in \mathcal{Z}(A)$  be a central idempotent of  $A$ . Then*

- 1)  $Ax$  is a two sided ideal of  $A$  if and only if  $x$  is central.
- 2)  $Ax = Ae$  if and only if  $x = e$ .

**Remark 2.5.** *Let  $A$  be a finite dimensional semisimple algebra over  $k$  and  $M$  be an  $A$ -module. A submodule of  $M$  will be called full isotopic submodule of  $M$  if and only if it is a sum of homogeneous components of  $M$ . It follows that  $N$  is a full isotopic submodule of  $M$  if and only if it is fixed by any  $A$ -endomorphism of  $M$ .*

**Theorem 2.6.** *A subcoalgebra  $C$  of  $H$  is invariant under the adjoint action if and only if its character  $d_C$  is central in  $H$ .*

*Proof.* Suppose that  $C$  is a subcoalgebra of  $H$  invariant under the adjoint action. Then  $C$  is a  $D(H)$ -submodule of  $H$ . The above description of the endomorphism ring of  $H$  shows that  $C$  is invariant under any  $D(H)$ -endomorphism of  $H$ . Thus  $C$  is a full isotopic submodule of  $H$ . Since  $\mathcal{F}$  is an isomorphism of  $D(H)$ -modules it follows that  $\mathcal{F}(C)$  is also a full isotopic submodule of  $H^*$ . Therefore  $\mathcal{F}(C) = \oplus_{i \in X} H^* E_i = H^*(\sum_{i \in X} E_i)$  for some set  $X$ . On the other hand by Remark 2.2 one has  $\mathcal{F}(C) = H^*(\sum_{d \in \text{Irr}(C)} \xi_d)$ . The previous lemma implies that  $\sum_{d \in \text{Irr}(C)} \xi_d = \sum_{i \in X} E_i$ . It follows that  $\sum_{d \in \text{Irr}(C)} \xi_d \in C(H)$  and therefore  $\sum_{d \in \text{Irr}(C)} \epsilon(d) d^* = \mathcal{F}^{-1}(\sum_{d \in \text{Irr}(C)} \xi_d)$  is central in  $H$ . But then  $d_C = S(\sum_{d \in \text{Irr}(C)} \epsilon(d) d^*)$  is also central in  $H$ .

Conversely suppose that the character  $d_C = \sum_{d \in \text{Irr}(C)} \epsilon(d) d$  is central in  $H$ . It follows that  $\mathcal{F}(d_C) \in C(H)$ . But  $\mathcal{F}(d_C) = \sum_{d \in \text{Irr}(C)} \xi_{d^*}$  is a central element in  $H^*$  and therefore a central character in  $C(H)$ . This implies that  $\sum_{d \in \text{Irr}(C)} \xi_{d^*} = \sum_{i \in X} E_i$ . Remark 2.2 implies that  $\mathcal{F}(C) = \sum_{i \in X} H^* E_i$ . Thus  $\mathcal{F}(C)$  is a  $D(H)$ -submodule of  $H^*$ . Since  $\mathcal{F}$

is an isomorphism of  $D(H)$ -modules this implies that also  $C$  is a  $D(H)$ -submodule of  $H$ . Thus  $C$  is invariant under the adjoint action.  $\square$

**Corollary 2.7.** *A two sided ideal  $I$  of  $H$  is co-normal if and only if its character as left (or right)  $H$ -module is central in  $H^*$ .*

*Proof.* By Proposition 2.3  $(H/I)^*$  is a normal subcoalgebra of  $H^*$ . Theorem 2.6 implies the conclusion by duality.  $\square$

### 3. GENERAL RESULTS

Let  $K$  be a Hopf subalgebra of  $H$ . Then the restriction functor from  $H$  to  $K$  defines an algebra map at the level of characters ring

$$\text{res}_K^H : C(H) \rightarrow C(K).$$

Similarly, the induction functor from  $K$  to  $H$  induces a linear map

$$\text{ind}_K^H : C(K) \rightarrow C(H).$$

For an irreducible character  $\alpha \in \text{Irr}(K)$  let  $f_\alpha$  be the central idempotent corresponding to  $\alpha$ . Similarly if  $\chi \in \text{Irr}(H)$  then  $e_\chi$  is the corresponding central idempotent in  $H$ . Consider the commutative algebra  $\mathcal{Z}(H) \cap K$  as a subalgebra of  $\mathcal{Z}(H)$  and  $\mathcal{Z}(K)$ . Then there are partition of characters  $\text{Irr}(H) = \bigsqcup_{i=1}^s \mathcal{A}_i$  and  $\text{Irr}(K) = \bigsqcup_{i=1}^s \mathcal{B}_i$  such that a basis of primitive idempotents for the above algebra is given by

$$m_i = \sum_{\chi \in \mathcal{A}_i} e_\chi = \sum_{\alpha \in \mathcal{B}_i} f_\alpha.$$

**Proposition 3.1.** *With the above notations*

$$\begin{aligned} \left( \sum_{\chi \in \mathcal{A}_i} \chi(1)\chi \right) \downarrow_K^H &= \frac{|H|}{|K|} \sum_{\alpha \in \mathcal{B}_i} \alpha(1)\alpha. \\ \left( \sum_{\alpha \in \mathcal{B}_i} \alpha(1)\alpha \right) \uparrow_K^H &= \sum_{\chi \in \mathcal{A}_i} \chi(1)\chi. \end{aligned}$$

*Proof.* By formula 1.1 it follows that

$$t_H = \frac{1}{|H|} \sum_{\chi \in \text{Irr}(H)} \chi(1)\chi.$$

This implies that  $\mathcal{F}_H(e_\chi) = \frac{\chi(1)}{|H|}\chi$ . Similarly  $\mathcal{F}_K(f_\alpha) = \frac{\alpha(1)}{|K|}\alpha$ . The first statement follows from the commutativity of the diagram from Proposition 1.5. Indeed,

$$\frac{1}{|K|} \sum_{\alpha \in \mathcal{A}_i} \alpha(1)\alpha = \mathcal{F}_K(m_i) = (\mathcal{F}_H(m_i)) \downarrow_K^H = \left(\frac{1}{|H|} \sum_{\chi \in \mathcal{A}_i} \chi(1)\chi\right) \downarrow_K^H.$$

For the second equality note that

$$\left(\sum_{\alpha \in \text{Irr}(K)} \alpha(1)\alpha\right) \uparrow_K^H = \sum_{\chi \in \text{Irr}(H)} \chi(1)\chi.$$

since both terms are the regular characters of  $H$ . Frobenius reciprocity applied to the first equality of this Proposition implies the second equality.  $\square$

**3.1. Image of Induction.** The following result is Proposition 2 of [1]. It shows that the image of the induction map is a two sided ideal in  $C(H)$ .

**Lemma 3.2.** *Let  $K$  be a Hopf algebra of a semisimple Hopf algebra. Let  $M$  be an  $H$  module and  $V$  a  $K$ -module. Then*

$$M \otimes V \uparrow_K^H = (M \downarrow_K^H \otimes V) \uparrow_K^H$$

and

$$V \uparrow_K^H \otimes M = (V \otimes M \downarrow_K^H) \uparrow_K^H$$

Let  $\epsilon_K$  the character of the trivial  $K$ -module. Let  $\epsilon \uparrow_K^H$  be the character corresponding to the trivial  $K$ -module induced to  $H$ .

**Proposition 3.3.** *Let  $K$  be a Hopf subalgebra of a semisimple Hopf algebra  $H$ . Then:*

- 1)  $\epsilon \uparrow_K^H C(H) \subset \text{Im}(\text{ind}_K^H)$
- 2)  $\mathcal{F}(K) \cap C(H) \subset \text{Im}(\text{ind}_K^H)$ .

*Proof.* Put  $V = k$ , the trivial  $K$ -module, in the second formula of the above lemma. In terms of the characters this becomes  $\chi \epsilon \uparrow_K^H = \chi \downarrow_K^H \uparrow_K^H$  for all  $\chi \in C(H)$ . This implies that  $\epsilon \uparrow_K^H C(H) \subset \text{Im}(\text{ind}_K^H)$ .

Using the above notations, since  $\mathcal{F}(K) \cap C(H) = \mathcal{F}(K \cap \mathcal{Z}(H))$  and  $\mathcal{F}$  is bijective it follows that  $\mathcal{F}(m_i)$  form a basis on  $C(H) \cap \mathcal{F}(K)$ . Proposition 3.1 shows that  $\mathcal{F}(m_i)$  are induced modules.  $\square$

#### 4. NORMAL HOPF SUBALGEBRAS

**4.1. Restriction to normal Hopf subalgebras.** Let  $H$  be a semisimple Hopf algebra over the algebraically closed field  $k$  and let  $K$  be a normal Hopf subalgebra of  $H$ . Define an equivalence relation on the set  $\text{Irr}(H)$  by  $\chi \sim \mu$  if and only if  $m_K(\chi \downarrow_K^H, \mu \downarrow_K^H) > 0$ . This is the equivalence relation  $r_{L^*, k}^{H^*}$  from [2] where  $L = H//K$ . It is proven that



$\chi \sim \mu$  if and only if  $\frac{\chi \downarrow_K^H}{\chi(1)} = \frac{\mu \downarrow_K^H}{\mu(1)}$  (see Theorem 4.3 of [2]). Thus the restriction of  $\chi$  and  $\mu$  to  $K$  either have the same irreducible constituents or they don't have common constituents at all.

The above equivalence relation determines an equivalence relation on the set of irreducible characters of  $K$ . Two irreducible  $K$ -characters  $\alpha$  and  $\beta$  are equivalent if and only if they are constituents of  $\chi \downarrow_K^H$  for some irreducible character  $\chi$  of  $H$ . Let  $\mathcal{C}_1, \dots, \mathcal{C}_{s'}$  be the equivalence classes of the equivalence relation defined on  $\text{Irr}(H)$ . Let  $\mathcal{D}_1, \dots, \mathcal{D}_{s'}$  be the corresponding equivalence classes of the equivalence relation defined on  $\text{Irr}(K)$ .

For each  $1 \leq i \leq s'$  put  $\alpha_i = \sum_{\alpha \in \mathcal{D}_i} \alpha(1)\alpha \in C(K)$  and  $a_i = \sum_{\chi \in \mathcal{C}_i} \chi(1)\chi \in C(H)$ .

The formulae from subsection 4.1 of [2] can be written as:

$$(4.1) \quad \chi \downarrow_K^H = \frac{\chi(1)}{\alpha_i(1)} \alpha_i$$

for all  $\chi \in \mathcal{C}_i$   
and

$$(4.2) \quad \alpha \uparrow_K^H = \frac{\alpha(1)}{a_i(1)} \frac{|H|}{|K|} a_i$$

for all  $\alpha \in \mathcal{D}_i$ .

**Remark 4.3.** Combining the above two formulae it can easily be seen that  $\alpha \uparrow_K^H \downarrow_K^H \uparrow_K^H = \frac{|H|}{|K|} \alpha \uparrow_K^H$  for all  $\alpha \in \text{Irr}(K)$ .

**Proposition 4.4.** Let  $K$  be a normal Hopf subalgebra of a semisimple Hopf algebra  $H$ . With the above notations one has

- 1)  $s = s'$ ,
- 2)  $\{\mathcal{C}_1, \dots, \mathcal{C}_s\} = \{\mathcal{A}_1, \dots, \mathcal{A}_s\}$ ,
- 3)  $\{\mathcal{D}_1, \dots, \mathcal{D}_s\} = \{\mathcal{B}_1, \dots, \mathcal{B}_s\}$ .

*Proof.* Let

$$p_i = \sum_{\alpha \in \mathcal{D}_i} f_\alpha$$

for all  $1 \leq i \leq s'$ . First we will show that  $p_i = \sum_{\chi \in \mathcal{C}_i} e_\chi$  for  $1 \leq i \leq s'$ .

Since both terms of the previous equality are idempotents it is enough to verify that any irreducible character of  $H$  takes the same value on both terms. Formula 4.1 shows that  $\chi(\sum_{\alpha \in \mathcal{D}_i} f_\alpha) = 0$  if  $\chi \notin \mathcal{C}_i$ . On

the other hand, if  $\chi \in \mathcal{C}_i$  then

$$\chi\left(\sum_{\alpha \in \mathcal{D}_i} f_\alpha\right) = \sum_{\alpha \in \mathcal{D}_i} m(\alpha, \chi \downarrow_K^H) \alpha(f_\alpha) = \sum_{\alpha \in \mathcal{D}_i} m(\alpha, \chi \downarrow_K^H) \alpha(1) = \chi(1)$$

and the equality is proved.

Thus each element  $p_i = \sum_{\alpha \in \mathcal{D}_i} f_\alpha$  belongs to  $\mathcal{Z}(H) \cap K$ . Since these elements are idempotent each of them is a sum of some of the primitive idempotents  $m_j$  of  $\mathcal{Z}(H) \cap K$ . But Proposition 3.1 shows that if  $\alpha \in \mathcal{B}_i$  and  $\beta \in \mathcal{B}_j$  with  $i \neq j$  then  $\alpha \not\sim \beta$ . Thus each element  $p_i$  coincide with one of the primitive idempotents  $m_j$  defined above. Since  $\sum_{i=1}^{s'} p_i = \sum_{i=1}^s m_i$  it also follows that any  $m_j$  coincide with one of the idempotents  $p_i$ . This fact implies all three equalities.  $\square$

**Proposition 4.5.** *If  $K$  is a normal Hopf subalgebra of a semisimple Hopf algebra  $H$  then  $\epsilon \uparrow_K^H C(H) = \text{Im}(\text{ind}_K^H) = \mathcal{F}(K) \cap C(H)$*

*Proof.* By Proposition 3.1 it follows that  $a_i \in \mathcal{F}(K) \cap C(H)$  for all  $1 \leq i \leq s$ . Since

$$\alpha \uparrow_K^H = \frac{\alpha(1)}{a_i(1)} a_i$$

for  $\alpha \in \mathcal{C}_i$  this shows that  $\text{Im}(\text{ind}_K^H) \subset \mathcal{F}(K) \cap C(H)$ . Then the second item of Proposition 3.3 implies  $\text{Im}(\text{ind}_K^H) = \mathcal{F}(K) \cap C(H)$ .

On the other hand using Lemma 3.2 and Remark 4.3 one has

$$(\alpha \uparrow_K^H) \epsilon \uparrow_K^H = (\alpha \uparrow_K^H \downarrow_K^H) \uparrow_K^H = \frac{|H|}{|K|} \alpha \uparrow_K^H$$

which shows that  $\text{Im}(\text{ind}_K^H) \subset \epsilon \uparrow_K^H C(H)$ . The first item of Proposition 3.3 implies  $\text{Im}(\text{ind}_K^H) = \epsilon \uparrow_K^H C(H)$ .  $\square$

**Proposition 4.6.** *Let  $K$  be a normal Hopf subalgebra of  $H$ . Then  $\mathcal{F}(K)$  and  $K^\perp$  are full isotopic submodules of  $H^*$  and the decomposition*

$$H^* = \mathcal{F}(K) \oplus K^\perp.$$

*of Proposition 1.3 is a decomposition of  $D(H)$ -modules.*

*Proof.* It can be checked directly that  $K$  is a submodule of  $H$  realized as before. Since  $\text{End}_{D(H)}(H) = C(H)$  and  $K$  is stabilized by any endomorphism of  $H$  it follows that  $K$  is a full isotopic submodule  $H$ . It also can be checked that  $K^\perp$  is stabilized by any endomorphism of  $H^*$ .  $\square$

4.1.1. *General Lemma.*

**Lemma 4.7.** *Suppose  $A$  is a finite dimensional semisimple algebra and  $M$  is a finite dimensional  $A$ -module. Moreover suppose that  $M = M_1 \oplus M_2$  where  $M_1$  and  $M_2$  are full isotopic submodules of  $M$ . Then the following are true:*

(1)

$$\text{End}_A(M) \cong \text{End}_A(M_1) \oplus \text{End}_A(M_2)$$

as  $k$ -algebras

(2)

$$\text{End}_A(M_1) = \{F \in \text{End}_A(M) \mid F(M_2) = 0\}$$

**Proposition 4.8.** *Let  $K$  be a normal Hopf subalgebra of  $H$ . Then:*

(1)

$$\text{End}_{D(H)}(K) = \{\chi \in C(H) \mid \sum a_1 \chi(Sa_2) \in K \text{ for all } a \in H\}$$

(2)

$$\text{End}_{D(H)}(K^\perp) = \{\chi \in C(H) \mid \sum x_1 \chi(S(x_2)) = 0 \text{ for all } x \in K\}$$

*Proof.* 1) The above Lemma for the decomposition of Proposition 4.6 implies that

$$\text{End}_{D(H)}(K) = \{\chi \in C(H) \mid K^\perp R_{S(\chi)} = 0\}$$

Since  $(fS(\chi))(a) = f(\sum a_1 \chi(S(a_2)))$  it follows that  $f(\sum a_1 \chi(S(a_2))) = 0$  for all  $f \in K^\perp$ . Thus  $\sum a_1 \chi(S(a_2)) \in K^{\perp\perp} = K$

2) From the above Lemma it also follows that  $\text{End}_{D(H)}(K^\perp) = \{\chi \in C(H) \mid |S(\chi) \rightharpoonup K = 0\}$  which is exactly the set mentioned above.  $\square$

Define

$$(4.9) \quad C_H^1(K) = \{\chi \in C(H) \mid \sum a_1 \chi(S(a_2)) \in K \text{ for all } a \in H\}$$

and

$$(4.10) \quad C_H^2(K) = \{\chi \in C(H) \mid \sum x_1 \chi(S(x_2)) = 0 \text{ for all } x \in K\}.$$

Then the above lemma implies that

$$C(H) = C_H^1(K) \oplus C_H^2(K)$$

as  $k$ -algebras.

**Remark 4.11.** *Note that  $C_H^2(K) = C(H) \cap K^\perp = \ker(\text{res}_K^H)$ . Indeed, if  $\chi \in C_H^2(K)$  then it follows  $\chi(Sx) = 0$  for all  $x \in K$ . The other inclusion is immediate.*

Recall the definition of  $\xi_d$  as the central primitive idempotent of  $H^*$  corresponding to the irreducible character  $d \in \text{irr}(H^*)$ .

**Lemma 4.12.** *Let  $K$  be a normal Hopf subalgebra of a semisimple Hopf algebra  $H$ . Then*

$$(4.13) \quad \epsilon \uparrow_K^H = \frac{|H|}{|K|} \sum_{d \in \text{Irr}(K^*)} \xi_d.$$

*Proof.* Since  $\Lambda_K$  is a central element of  $H$  it follows that

$$\Lambda_K = \sum_{\chi \in \mathcal{A}} e_\chi$$

for some subset  $\mathcal{A} \subset \text{Irr}(H)$ . Applying  $\mathcal{F}$  to this equality one gets that:

$$\mathcal{F}(\Lambda_K) = \frac{1}{|H|} \sum_{\chi \in \mathcal{A}} \chi(1)\chi.$$

On the other hand Proposition 3.1 implies that

$$\left( \sum_{\chi \in \mathcal{A}} \chi(1)\chi \right) \downarrow_K^H = \frac{|H|}{|K|} \epsilon_K.$$

The same Proposition implies that all the other characters of  $H$  that are not contained in  $\mathcal{A}$  do not contain  $\epsilon_K$  when restricted to  $K$ . Frobenius reciprocity then gives

$$\epsilon \uparrow_K^H = \sum_{\chi \in \text{Irr}(H)} m_K(\chi \downarrow_K^H, \epsilon_K) = \sum_{\chi \in \mathcal{A}} \chi(1)\chi.$$

Thus  $\epsilon \uparrow_K^H = |H| \mathcal{F}(\Lambda_K)$ .

On the other hand, since  $\Lambda_K = \frac{1}{|K|} \sum_{d \in \text{Irr}(K^*)} \epsilon(d)d$

and  $\mathcal{F}(d) = \frac{\xi_{d^*}}{\epsilon(d)}$  one can write

$$\mathcal{F}(\Lambda_K) = \frac{1}{|K|} \sum_{d \in \text{Irr}(K^*)} \xi_{d^*} = \frac{1}{|K|} \sum_{d \in \text{Irr}(K^*)} \xi_d.$$

Thus

$$\epsilon \uparrow_K^H = \frac{|H|}{|K|} \sum_{d \in \text{Irr}(K^*)} \xi_d.$$

□

Since  $H$  is cosemisimple it is the sum of its simple subcoalgebras. These subcoalgebras are in bijective correspondence with the irreducible  $H^*$ -characters [5]. Since  $k$  is algebraically closed, if  $d \in \text{Irr}(H^*)$  then

its associated subcoalgebra  $C_d$  is a matrix coalgebra. This means that  $C_d$  has a  $k$ -basis  $\{x_{ij}^d\}_{1 \leq i, j \leq q}$  such that

$$(4.14) \quad \Delta(x_{ij}^d) = \sum_{l=1}^q x_{il}^d \otimes x_{lj}^d$$

and  $\epsilon(x_{ij}) = \delta_{i,j}$  for all  $1 \leq i, j \leq q$ . Moreover  $\epsilon(d) = q$  and the irreducible character  $d$  is given by  $d = \sum_{i=1}^q x_{ii}^d$ . Therefore one can write

$$H = \oplus_{d \in \text{Irr}(H^*)} C_d$$

which shows that  $x_{ij}^d$  form a  $k$ -basis for  $H$ . Clearly  $\text{Irr}(K^*) \subset \text{Irr}(H^*)$  and

$$H = \oplus_{d \in \text{Irr}(K^*)} C_d.$$

On the other hand, by its definition it follows that  $\xi_d(x_{ij}^{d'}) = \delta_{d, d'} \delta_{i,j}$  for all  $d, d' \in \text{Irr}(H)$ .

**Remark 4.15.** Note that the condition  $\chi \in C_H^1(K)$  is equivalent to  $\chi(x_{ij}^d) = 0$  for all  $d \in \text{Irr}(H^*) \setminus \text{Irr}(K^*)$ . This can be seen from formula 4.14.

**Theorem 4.16.** Let  $K$  be a normal Hopf subalgebra of  $H$  and  $\text{ind}_K^H : C(K) \rightarrow C(H)$  be the map induced by the induction functor. Then

$$C_H^1(K) = \text{im}(\text{ind}_K^H).$$

*Proof.* Formula 4.13 together with 4.14 show that  $\epsilon \uparrow_K^H \chi \in C_H^1(K)$  for all  $\chi \in C(H)$ . Indeed for a basis element  $a = x_{uv}^d$  the identity from the definition 4.9 of  $C_H^1(K)$  is verified since the element  $\sum a_1 \chi(S(a_2))$  is zero if  $d \notin \text{Irr}(K^*)$ . Obviously  $\sum a_1 \chi(S(a_2)) \in K$  if  $d \in \text{Irr}(K^*)$ . Proposition 4.5 implies  $\text{im}(\text{ind}_K^H) = \epsilon \uparrow_K^H C(H) \subset C_H^1(K)$ . On the other hand it will be checked that if  $\chi \in C_H^1(K)$  then  $\chi \epsilon \uparrow_K^H = \frac{|H|}{|K|} \chi$  which shows the other inclusion. The equality  $\chi \epsilon \uparrow_K^H = \frac{|H|}{|K|} \chi$  is verified by evaluating both terms on the basis elements  $x_{ij}^d$  of  $H$ . If  $d \notin \text{Irr}(K^*)$  then both evaluations are zero by Remark 4.15. On the other hand if  $d \in \text{Irr}(K^*)$  then the evaluations are equal from the definition of  $\xi_d$  and the formula for  $\epsilon \uparrow_K^H$  given in Lemma 4.13.  $\square$

**Corollary 4.17.** Let  $K$  be a normal Hopf subalgebra of a finite dimensional semisimple Hopf algebra  $H$ . Then  $\text{im}(\text{ind}_K^H)$  and  $\text{im}(\text{res}_K^H)$  are isomorphic as  $k$ -algebras.

*Proof.* Since  $C_H^2(K) = \ker(\text{res}_K^H)$  it follows that the image of the restriction map is isomorphic to  $C(H)/C_H^2(K) \cong C_H^1(K)$  as  $k$ -algebras.  $\square$

**Theorem 4.18.** *Let  $K$  be a normal Hopf subalgebra of a finite dimensional Hopf algebra  $H$ . The image of the restriction map  $\text{res}_K^H$  is isomorphic as  $k$ -algebras with the subalgebra of  $C(K)$  spanned by*

$$\{f \in K^* | f(\sum a_1 x S(a_2)) = \epsilon(a)f(x) \text{ for all } a \in A, x \in K\}.$$

*Proof.* Clearly the image of the restriction lies in the space

$$\{f \in K^* | f(\sum a_1 x S(a_2)) = \epsilon(a)f(x) \text{ for all } a \in A, x \in K\}.$$

Let  $v$  be a linear functional of this space. It follows that  $v \rightharpoonup$  is an endomorphism of  $K$  as  $D(H)$ -module. Indeed for all  $a \in H$  and  $x \in K$  one has that:

$$\begin{aligned} v \rightharpoonup (a.x) &= \sum v \rightharpoonup a_1 x S(a_2) = \sum a_1 x_1 S(a_4) v(a_2 x_2 S(a_3)) = \\ &= \sum a_1 x_1 S(a_2) v(x_2) = a.(v \rightharpoonup x) \end{aligned}$$

It also can be checked that  $v \rightharpoonup (f.x) = f.(v \rightharpoonup x)$  for all  $x \in K$  and  $f \in H^*$ . Indeed,

$$v \rightharpoonup (f.x) = v \rightharpoonup \sum f(Sx_1)x_2 = \sum f(Sx_1)x_2 v(x_3).$$

On the other hand,

$$f.(v \rightharpoonup x) = \sum f.v(x_2)x_1 = \sum f(Sx_1)x_2 v(x_3).$$

Since  $\text{End}_{D(H)}(H) = C(H)$  and  $K$  is a full isotopic  $D(H)$ -submodule of  $H$  it follows that there is  $\chi \in C(H)$  such that  $v \rightharpoonup = \chi \rightharpoonup$  on  $K$ . This implies that  $v = \chi \downarrow_K^H$ .  $\square$

## REFERENCES

1. S. Burciu, On some representations of the Drinfeld double, J. Algebra **296** (2006), 480504.
2. ———, Coset Decomposition For Semisimple Hopf Algebras, arXiv:0712.1719 (2007).
3. L. Kadison and S. Burciu, Semisimple Hopf algebras and their depth two Hopf subalgebras, preprint (2008).
4. Y. Kashina, Y. Sommerhäuser, and Y. Zhu, Higher Frobenius-Schur indicators, vol. 181, Mem. Am. Math. Soc., Am. Math. Soc., Providence, RI, 2006.
5. R. G. Larson, Characters of Hopf algebras, J. Algebra (1971), 352–368.
6. R. G. Larson and D. E. Radford, Finite dimensional cosemisimple Hopf Algebras in characteristic zero are semisimple, J. Algebra **117** (1988), 267–289.
7. R.G. Larson and M.E. Sweedler, An associative orthogonal bilinear form for Hopf algebras, Amer. J. Math. **91** (1969), no. 7, 7593.
8. A. Masuoka, Semisimple Hopf algebras of dimension 2p, Comm. Algebra **23** (1995), no. 5, 1931–1940.
9. S. Montgomery, Hopf algebras and their actions on rings, vol. 82, 2nd revised printing, Reg. Conf. Ser. Math, Am. Math. Soc, Providence, 1997.

10. S. Natale, Semisolvability of semisimple Hopf algebras of low dimension, no. 186, Mem. Am. Math. Soc., Am. Math. Soc., Providence, RI, 2007.
11. W. D. Nichols and M. B. Richmond, The Grothendieck group of a Hopf algebra, J. Pure and Appl. Algebra **106** (1996), 297–306.
12. ———, The Grothendieck group of a Hopf algebra, I, Comm. Algebra **26** (1998), 1081–1095.
13. Y. Zhu, Hopf algebras of prime dimension., Int. Math. Res. Not. **1** (1994), 53–59.
14. ———, A commuting pair in Hopf algebras, Proc. Am. Math. Soc. **125** (1997), 2847–2851.

INST. OF MATH. “SIMION STOILOW” OF THE ROMANIAN ACADEMY P.O. BOX  
1-764, RO-014700, BUCHAREST, ROMANIA  
*E-mail address:* `sebastian.burciu@imar.ro`